XXX. *Geometrical Solutions of three celebrated Aſtronomical Problems, by the late Dr.* Henry Pemberton, *F. R. S. Communicated by* Matthew Raper, *Eſq; F. R. S.*

LEMMA.

Read June 4, 1772.

TO form a triangle with two given ſides, that the rectangle under the ſine of the angle contained by the two given ſides, and the tangent of the angle oppoſite to the leſſer of the given ſides, ſhall be the greateſt that can be.

Let [TAB. XII. Fig. 1.] the two given ſides be equal to AB and AC: round the center A, with the interval AC, deſcribe the circle CDE, and produce BA to E; take BF a mean proportional between BE and BC, and erect the perpendicular FG, and complete the triangle AGB.

Here the ſine of BAG is to the radius, as FG to AG; and the tangent of ABG to the radius, as FG to FB: therefore, the rectangle under the ſine of BAG and the tangent of ABG is to the ſquare of the

Fig. 1.

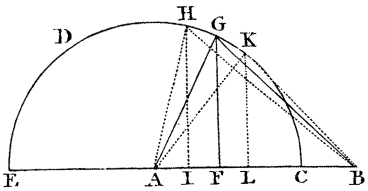
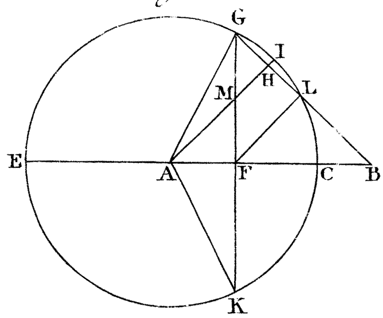


Fig. 2.



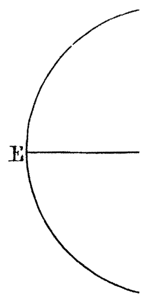


Fig. 4.

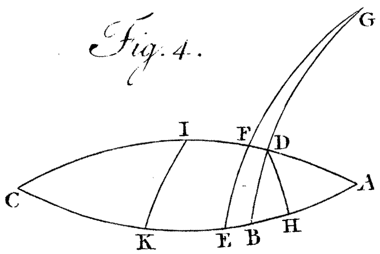


Fig. 5.

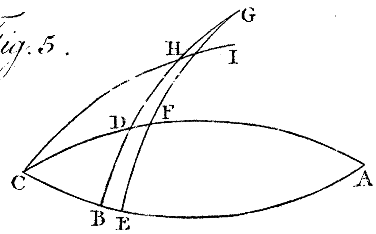
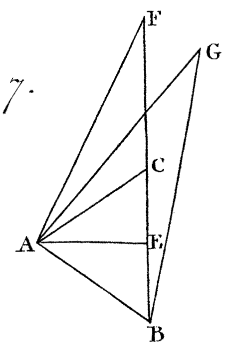


Fig. 6.

Fig. 7.



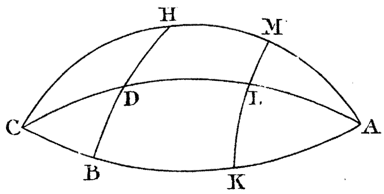


Fig. 8.

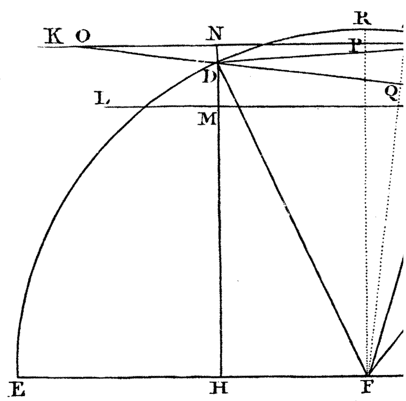


Fig. 9

Fig. 3.

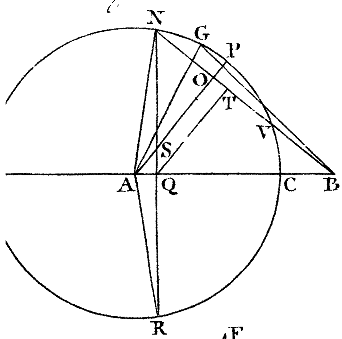
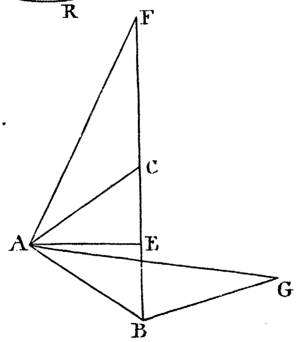


Fig. 6.



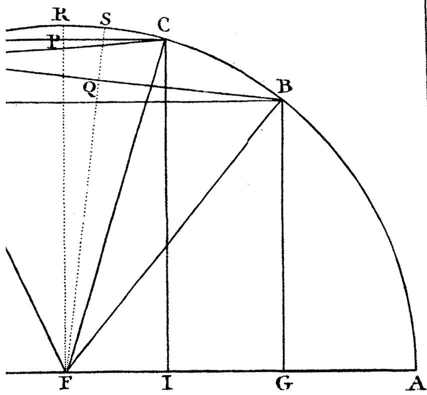


Fig. 9.

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the radius, as the ſquare of FG, or the rectangle EFC, to the rectangle under AG (or AC) and FB. But, EB being to BF as BF to BC, by converſion, EB is to EF as BF to FC, and alſo, by taking the difference of the antecedents and of the conſequents, EF is to twice AF as BF to FC; and twice AFB is equal to EFC.

Now, let the triangle BAH be formed, where the angle BAH is greater than BAG. Here, the perpendicular HI being drawn, the rectangle under the ſine of BAH and the tangent of ABH will be to the ſquare of the radius, as the rectangle EIC to the rectangle under AC, IB. But IF is to FB as 2AFI to 2AFB, or EFC; and 2AFI is greater than $AF^2 - AI^2$; alſo $AF^2 - AI^2$ together with EFC, is equal to EIC; therefore, by compoſition, the ratio of IB to BF is greater than that of EIC to EFC; and the ratio of $AC \times IB$ to $AC \times FB$ greater than that of EIC to EFC: alſo, by permutation, the ratio of $AC \times IB$ to EIC greater than the ratio of $AC \times FB$ to EFC. But the firſt of theſe ratios is the ſame with that of the ſquare of the radius to the rectangle under the ſine of BAH and the tangent of ABH; and the latter is the ſame with that of the ſquare of the radius to the rectangle under the ſine of BAG and the tangent of ABG; therefore, the latter of theſe two rectangles is greater than the other.

Again, let the triangle BAK be formed, with the angle BAK leſs than BAG, and the perpendicular KL be drawn. Then the rectangle under the ſine of BAK and the tangent of ABK is to the ſquare of the radius, as the ſquare of KL to the rectangle under

AC, BL. Here, FL being to FB as 2 AFL to 2 AFB or EFC, and 2 AFL less than $AL^2 - AF^2$, by conversion, the ratio of LB to FB will be greater than the ratio of ELC to EFC; therefore, as before, the rectangle under the sine of BAG and the tangent of ABG is greater than that under the sine of BAK and the tangent of ABK.

COROLLARY I.

BF is equal to the tangent of the circle from the point B; therefore, BF is the tangent, and AB the secant, to the radius AC, of the angle, whose cosine is to the radius as AC to AB. Therefore, AF is the tangent, to the same radius, of half the complement of that angle; and AF is also the cosine of the angle BAG to this radius.

COROL. 2.

The sine of the angle composed of the complement of AGB, and twice the complement of ABG, is equal to three times the sine of the complement of AGB. Let fall the perpendicular AH (Fig. 2.), cutting the circle in I; continue GF to K, and draw AK. Then $BF^2 = EBC = GBL$. Therefore, GB : BF :: BF : BL, and the triangles GBF, FBL are similar. Consequently FL is perpendicular to GB, and parallel to AH; whence GH being equal to HL, GM is equal to MF, and MK equal to three times GM.

Now, the arc IK = 2 IC + GI; and the angle IAK = 2 IAC + GAI; also GM is to MK as the

the ſine of the arc GI to the ſine of the arc IK, that is, as the ſine of the angle GAI to the ſine of the angle IAK. Therefore, the ſine of the angle IAK (=2IAC + GAI) is equal to three times the ſine of the angle GAI; but GAI is the complement of AGB, and IAC the complement of ABG.

Corol. 3.

If (Fig. 3.) any line BN be drawn to divide the angle ABG, and AN be joined, alſo AO be drawn perpendicular to BN, and continued to the circle in P, the ſine of the angle compoſed of NAP and 2PAC will be leſs than three times the ſine of the angle NAP. Draw NQR perpendicular to AB, cutting AP in S; join AR, and draw QT perpendicular to BN, and parallel to AO; then BQq = NBT. But BQq is greater than the rectangle EBC, that is, greater than the rectangle NBV, under the two ſegments of the line BN drawn from B, to cut the circle in N and V: therefore, TB is greater than VB, and NO greater than OT. Conſequently NS is greater than SQ. Hence RS is leſs than three times NS; and therefore, the ſine of the angle PAR (= NAP + 2PAC) is leſs than three times the ſine of NAP.

Problem

PROBLEM I.

To find in the ecliptic the point of longeſt *aſcenſion.*

ANALYSIS.

Let (Fig. 4.) ABC be the equator, ADC the ecliptic, BD the ſituation of the horizon, when D is the point of longeſt aſcenſion. Let EFG be another ſituation of the horizon. Then the ratio of the ſine of EB to the ſine of FD is compounded of the ratio of the ſine of BG to the ſine of GD, and of the ratio of the ſine of AE to the ſine of AF; but the angles B and E being equal, the arcs EG, GB together make a ſemicircle; and, by the approach of EG towards GB, the ultimate magnitude of BG will be a quadrant, and the ultimate ratio of EB to FD will be compounded of the ratio of the radius to the ſine of DG (that is, the coſine of BD) and of the ratio of the ſine of AB to the ſine of AD. Draw the arc DH perpendicular to AB. Then, in the triangle BDH, the radius is to the coſine of BD, as the tangent of the angle BDH to the cotangent of HBD. Alſo, in the triangle BDA, the ſine of AB is to the ſine of AD as the ſine of the angle BDA (or BDC) to the ſine of ABD; therefore, the ultimate ratio of BE to DF is compounded of the ratio of the tangent of BDH to the cotangent of ABD, and of the ratio of the ſine of BDC to the ſine of ABD; which two ratios compound that of the rectangle under the tangent of BDH and the ſine of BDC to the rectangle under the cotangent and the ſine of the given angle ABD.

 But

But, when D is the point of longeſt aſcenſion, the ratio of BE to DF is the greateſt that can be; therefore, then the ratio of the rectangle under the tangent of BDH and the ſine of BDC to the given rectangle under the cotangent and ſine of the given angle ABD muſt be the greateſt that can be; and conſequently, the rectangle under the tangent of BDH, and the ſine of BDC, muſt be the greateſt that can be.

In the triangle BDA, the ſine of BDH is to the ſine of HDA, as the coſine of ABD to the coſine of BAD. Now, in the preceding lemma, let the angle BAG of the triangle AGB be equal to the ſpherical angle BDC: then will the ſum of the angles ABG, AGB be equal to the ſpherical angle BDA. And, if AG in the triangle AGB, be to AB as the coſine of the ſpherical angle DBA to the coſine of DAB, that is, as the ſine of BDH to the ſine of HDA, the angle ABG, in the triangle, will be equal to the ſpherical angle BDH; and the angle AGB, in the triangle, equal to the ſpherical angle HDA. Therefore, by the firſt corollary of the lemma, that the rectangle under the tangent of the ſpherical angle BDH and the ſine of BDC be the greateſt that can be, the coſine of BDC muſt be equal to the tangent of half the complement of the angle, whoſe coſine is to the radius, as AG to AB, in the triangle, or as the coſine of the ſpherical angle ABD to the coſine of the ſpherical angle BAD.

If IK be the ſituation of the horizon, when the ſolſtitial point is aſcending, in the quadrantal triangle AIK, the coſine of KIC is to the radius as the coſine of IKA (= DBA) to the coſine of IAK. Therefore,

fore, the cosine of BDC, when D is the point of longest ascension, is equal to the tangent of half the complement of the angle, which the ecliptic makes with the horizon, when the solstitial point is ascending.

But, the sine of the angle composed of DAB, and twice ABD, must be less than three times the sine of the angle BAD. In the spherical triangle ABD, the angles BAD, ABD together exceed the external angle BDC. Therefore, in the third corollary of the lemma, let the angle BAN be equal to the sum of the spherical angles BAD, ABD: but here, AN is to AB as the cosine of the spherical angle ABD to the cosine of BAD; and AN is also to AB as the sine of ABN to the sine of ANB, that is, as the cosine of BAP to the cosine of NAP; consequently, since the angle BAN is equal to the sum of the spherical angles BAD, ABD, the angle NAP is equal to the spherical angle BAD, and the angle BAP equal to the spherical angle ABD; but the sine of the angle composed of NAP and twice PAB is less than three times the sine of NAP; therefore, the sine of the angle composed of the spherical angle BAD and 2 ABD will be less than three times the sine of the angle BAD; otherwise no such triangle DBA, as is here required, can take place, but the point A will be the point of longest ascension.

If the sine of the angle A be greater than one third of the radius, the point A can never be the point of longest ascension; but when the sine of this angle is less, the angle compounded of BAD and twice ABD, may be greater or less than a quadrant; and

and therefore, the magnitude of the angle ABD, that A be the point of longest ascension, is confined within two limits, of which the double of one added to the angle A, as much exceeds a quadrant, as the double of the other added to that angle falls short of it; therefore, double the sum of those two angles, together with twice A, makes a semicircle; and the single sum of those two angles added to A makes a quadrant.

PROBLEM II.

To find when the arc of the ecliptic differs most from its oblique ascension.

ANALYSIS.

If (Fig. 5.) BD be the situation of the horizon, when CD differs most from CB, as before, the ultimate ratio of BE to DF will be compounded of the ratio of the radius to the sine of DG (or the cosine of DB) and of the ratio of the sine of CB to the sine of CD: but, when CD differs most from CB, BE and DF are ultimately equal; therefore, then the cosine of BD is to the radius as the sine of CB to the sine of CD.

Draw the arc CHI of a great circle, that DH be equal to DB; then, BH being double BD, half the sine of BH is to the sine of BD or DH, as the cosine of BD to the radius; therefore, half the sine of BH is to the sine of DH as the sine of CB to the sine of CD; but the sine of the angle BCH is to the sine of BH as the sine of the angle CHB to the

 sine

ſine of CB; whence, by equality, half the ſine of BCH is to the ſine of DH as the ſine of CHB to the ſine of CD: but as the ſine of CHB to the ſine of CD, ſo, in the triangle CHD, is the ſine of DCH to the ſine of HD: conſequently, the ſine of DCH is equal to half the ſine of BCH. Hence, the difference of the angles BCH, DCH being given, thoſe angles are given, and the arc CHI is given by poſition.

Moreover, in the triangle BCH, the baſe BH being biſected by the arc CD, the ſine of the angle CHD is to the ſine of the given angle CBD, as the ſine of the given angle HCD to the ſine of the given angle BCD; therefore, the angle CHB is given; in ſo much, that in the triangle CBH all the angles are given.

The ſum of the ſines of the angles BCH, DCH is to the difference of their ſines, as the tangent of half the ſum of thoſe angles to the tangent of half their difference; therefore, the tangent of half the ſum of BCH, DCH is three times the tangent of half BCD.

In (Fig. 6.) the iſoſceles triangle ABC, let the angle BAC be equal to the ſpherical angle BCD, and let AE be perpendicular to BC; alſo, CF being taken equal to CB, join AF: then EF is equal to three times EB; and as EF to EB, ſo is the tangent of the angle EAF to the tangent of EAB; but EAB is equal to half the ſpherical angle BCD: therefore, the angle EAF is equal to half the ſum of the ſpherical angles BCD, BCH; and conſequently, the angle CAF equal to the ſpherical angle DCH. Here, AF is to CF as the ſine of the angle ACF to the ſine of C A F; and C B is to A B as the ſine of the angle B A C to the ſine of A C B: therefore, C F being equal to C B, and the ſine of A C F to the ſine of A C B, by equality, A F is to A B as the ſine of the angle B A C to the ſine of C A F, that is, as the ſine of the ſpherical angle B C D to the ſine of the ſpherical angle D C H.

Let (Fig. 7.) the triangle A G B have the angle A B G equal to the ſpherical angle C B D, and the ſide A G equal to A F. Then, A G is to A B as the ſine of the ſpherical angle B C D to the ſine of the ſpherical angle D C H, that is, as the ſine of the ſpherical angle C B H to the ſine of the ſpherical angle C H B: but A G is to A B alſo as the ſine of the angle A B G to the ſine of A G B; therefore, the angle A B G being equal to the ſpherical angle C B H, the angle A G B is equal to the ſpherical angle C H B: and moreover, when the angle A B G is greater than A B F, that is, when the ſpherical angle C B H is greater than the complement of half B C D, the three angles A B G, A G B and B A C together exceed two right.

Hence, (Fig. 8.) towards the equinoctial point C, where the angle C B D is obtuſe, a ſituation of the horizon, as B D, may always be found, wherein C D more exceeds C B than in any other ſituation: and when the acute angle D B A is greater than the complement of half B C D, another ſituation of the horizon, as K L M, may be found, toward the other equinoctial point A, wherein the arc of the ecliptic C K will be leſs than the arc of the equator, and their difference be greater than in any other ſituation. But, if the angle D B A be not greater than the com-

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plement of half BCD, the arc of the ecliptic, between C and the horizon, will never be lefs than the arc of the equator, between the fame point C and the horizon.

In the two fituations of the horizon, the angles CHB and KMA are equal.

SCHOLIUM I.

To find the point in the ecliptic, where the arc of the ecliptic moft exceeds the right afcenfion, is a known problem: that point is, where the cofine of the declination is a mean proportional between the radius and the cofine of the greateft declination.

In the preceding figure, fuppofing the angle CBD to be right, then, becaufe when CD moft exceeds CB, the cofine of BD is to the radius as the fine of CB to the fine of CD, and, in the triangle CBD, the fine of CB is to the fine of CD as the fine of the angle CDB to the radius, alfo the fine of CDB is to the radius as the cofine of BCD to the cofine of BD; therefore, the cofine of BD is to the radius as the cofine of the angle BCD to the cofine of the fame BD, and the cofine of BD is a mean proportional between the radius and the cofine of BCD.

SCHOLIUM 2.

In any given declination of the Sun, to find when the azimuth moft exceeds the angle which meafures the time from noon, is a problem analogous to the preceding.

Dr.

PROBLEM III.

*The tropic found, by Dr. Halley's method *, without any consideration of the parabola.*

The obſervations are ſuppoſed to give the proportions between the differences of the ſines of three declinations of the Sun near the tropic; but the ſine of the Sun's place is in a given proportion to the ſine of the declination; therefore, the ſame obſervations give equally the proportion between the differences of the ſines of the Sun's place, in each obſervation.

Now (Fig. 9.), let ACE be the ecliptic, AE its diameter between ♈ and ♎, and its center F; let B, C, D be three places of the Sun; BG, CI, DH the ſines of thoſe places reſpectively. Draw CK, BL parallel to AE, which may meet HD, in N and M. Then, by the obſervations, the ratio of DM to DN is given. Therefore, if BD be drawn to meet KL in O, the ratio of BD to OD is given; and the ratio of BD to DC is alſo given, they being the chords of the given angles BFD, CFD: hence the ratio of CD to DO, in the triangle CDO, is given; and conſequently, the angle COD will be given: which angle is the diſtance of the tropic from the middle point of the ecliptic between B and D: for, FPR being perpendicular to OC, and FQS perpendicular to DB, the angle QFP is equal to QOP, the points O, P, Q, F, being in a circle.

* Vide Philoſophical Tranſactions, N° 215.

THE

THE CALCULATION.

$$\left.\begin{matrix} DN : DM \\ \text{ſ. } \tfrac{1}{2} BFD : \text{ſ. } \tfrac{1}{2} CFD \end{matrix}\right\} :: \text{rad.} : \text{t.} \angle \chi$$

$$\text{rad.} : \text{t.} \angle \overline{\chi \backsim 45^\circ} :: \text{t.} \tfrac{1}{4} BFC : \text{t.} \frac{COD \backsim DCO}{2}$$

If $\chi > 45^\circ$, $\angle COD > DCO$

And

if $\chi \angle 45^\circ$, $\angle COD < DCO$.

If the intervals between the obſervations are ſo ſmall, that the ſines differ not much from the arches, the arches BC, CD may be counted in time, and the calculation may be abbreviated thus:

$DM : DN ::$ arc. $BD : Z$ (for DO)
$DC + Z : 2\,DC :: \tfrac{1}{4} BC : SR.$

Or,

$DM \times DC + DN \times BD : DM \times DC :: \tfrac{1}{2} BC : SR.$

XXXI. *On*

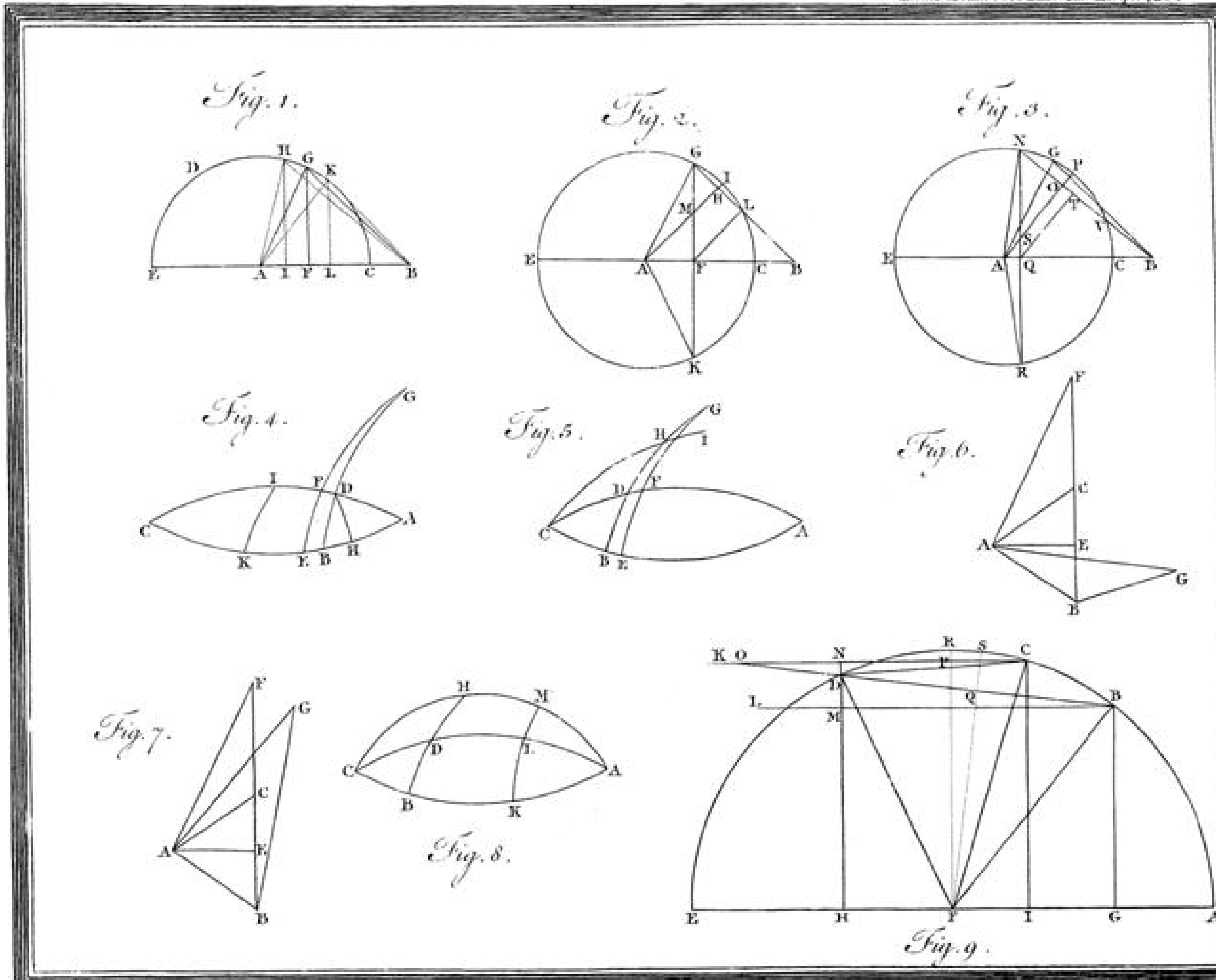
Fig. 1.
Fig. 2.
Fig. 3.
Fig. 4.
Fig. 5.
Fig. 6.
Fig. 7.
Fig. 8.
Fig. 9.